COMPUTATION OF EXTENSIVE FORM EQUILIBRIA IN SEQUENCE FORM GAMES

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Abstract

This paper presents a modified version of a sequence form representation of a two-person extensive-form game to compute an extensive-form perfect equilibrium. The modified version enables us to compute equilibrium of a given game by solving a linear complementarity program in general sum-games and by solving a linear program in zero-sum games.

1 Introduction

Following the work of Lemke and J. T. Howson (1964), significant works have examined the computational aspects of normal-form solution concepts. Using the algorithm of Lemke and J. T. Howson (1964), we can efficiently compute Nash equilibrium in zero-sum bi-matrix games. Despite the improvement this approach offer over normal-form games, the computation of extensive-form game solutions has not received much attention. Normal-form games are often too abstract and incapable of capturing and describing economic phenomena. For example, the entry-deterrence game in normal-form representation omits details pertaining to the timings and information available to the firms, while its extensive-form counterpart does not. Thus, it will be important for us to extend our understanding to equilibrium computation on extensive-form games if we are to understand how economic agents make decisions and identify the consequences of the interactions among agents. In the current paper, we present a modified version of sequence-form representation of a two-person extensive-form game to compute at least one extensive-form (trembling-hand) perfect equilibrium.

2 Related Literature

A sequence-form representation of an extensive-form game was first introduced by von Stengel (1996) who demonstrated that each Nash equilibrium of a general two-person game is a solution to a linear complementarity problem.(henceforth referred to as LCP). Especially, in zero-sum cases, a Nash equilibrium is shown to be a solution to a linear program. As a result, the equilibrium can be computed efficiently in zero-sum games.[1] Within his research, von Stengel, van den Elzen, and Talman (2002) applies the sequence-form representation to find

[1] There are polynomial time algorithm for linear programs. See Khachiyan (1979, 1980) or Karmarkar (1984).

a normal-form perfect equilibrium[2]. This work proposes an extensive-form counterpart of the algorithm of Lemke and J. T. Howson (1964). For a further refinement of Nash, Miltersen and Sørensen (2010) applies Lemke's algorithm to compute a quasi-perfect equilibrium to extensive-form games.[3] In this study, a modified version of sequence form is introduced that allows trembles at each information set. This work highlights that a quasi-perfect equilibrium is the limit equilibrium as the probability of trembles vanish. The first algorithm to compute an extensive-form perfect equilibrium is suggested by Gatti and Iuliano (2011). In addition to the tremble used in Miltersen and Sørensen (2010), the paper presents another type of tremble that is referred to as a dual tremble. While the tremble of Miltersen and Sørensen (2010) describes the possibility of an opponent making mistakes, the dual tremble make it possible to take into account the player's own mistakes. The dual tremble is applied both to the payoff and to the constraint matrices in sequence form games. The authors construct an ϵ-perturbed game using these two types of trembles, and demonstrate that the complexity of computing an extensive-form perfect equilibrium in general two person games is in PPAD class. The present paper also provides a method of computing an extensive-form perfect equilibrium. However, this paper deviates from the work of Gatti and Iuliano (2011) in various aspects. First, instead of using primal-dual perturbation, we directly use Selten's tremble to model ϵ-perturbed games. By doing so, we can avoid modifying or parameterizing payoff matrices, and can simplify the program. Second, we provide a method to recover a limit equilibrium from a computed equilibrium of an ϵ-perturbed game. Third, Gatti and Iuliano (2011) focuses on the development of an algorithm that could solve general sum games. One may expect that zero-sum games can be solvable more efficiently, and this paper supports this assumption.

Sequence form is widely used to compute extensive-form equilibrium for two main reasons: bilinearity structure of expected payoffs and its succinct size. However, it has a disadvantage in terms of its ability to describe dynamic choices: Given the other players' choices (or realization plans), the player's problem collapses to a static problem in sequence-form representation. This partially explains why introducing trembles directly to the sequence form as in Miltersen and Sørensen (2010) computes normal-form perfect equilibria rather than extensive-form perfect equilibria. Even though there is a superficial similarity between extensive-form perfection and normal-form perfection, they are different solution concepts. Mertens (1995) presents an example of a situation in which extensive-form perfection and normal-form perfection do not coincide. The figure below briefly shows the relationship among different solution concepts.

[2]Trembling hand perfect equilibrium. It in normal form games does not have to be a perfect equilibrium in extensive form games (see van Damme (1987))

[3]Quasi-perfect equilibrium due to van Damme (1984) is a refinement of normal form perfect equilibrium.

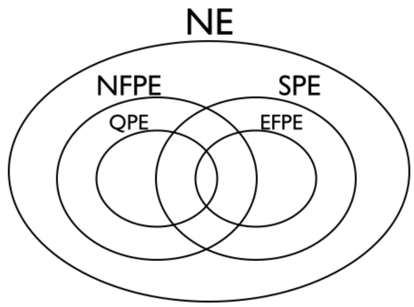

Figure 1: Relation among solution concepts NE: Nash equilibrium
NE: Nash equilibrium, NFPE: Normal form perfect equilibrium, SPE: Subgame perfect equilibrium, QPE: Quasi-perfect equilibrium, EFPE: Extensive form perfect equilibrium

In this paper, we will use a modified version of sequence form to compute extensive-form perfect equilibrium, one of the most renowned refinement of Nash equilibrium. We develop a program that solves equilibrium in ϵ-perturbed games and provide a conversion formula to compute its limit equilibrium. Since the conversion formula can be applied to games with any type of payoff structure, we confirm that an extensive form perfect equilibrium of two-person zero-sum extensive-form games with perfect recall can be solvable in time polynomial to its size. The plan of this paper is as follows: Section 3 defines the sequence form representation of von Stengel (1996) and introduces modified version of it. Section 4 provides a method for solving an extensive-form perfect equilibrium using the modified form of von Stengel's sequence form.

3 Modeling

In this section, we briefly introduce the sequence-form representation of von Stengel (1996) and then introduce a modified version of it that is subsequently used to compute extensive-form perfect equilibria.

3.1 The sequence form representation

We will briefly review the sequence form representation of von Stengel (1996). Readers who are familiar with sequence form may skip this subsection 3.1.

A common way to compute an equilibrium in extensive form game is to convert it into a normal form game so that we can apply tools developed in normal form games. However, there can be two major issues of this conversion. First, it may ignore detailed strategic descriptions of the original game. Even though an extensive form game has its unique normal form representation, its normal form counterpart abstracts from chronology specified in the original game. Secondly, a computational efficiency issue rises: the size of normal form representation blows up as the number of actions for each player increases. This blow-up can be avoided by using sequence form representation. To describe a sequence form representation of an extensive form game, we will take the original notations of von Stengel (1996). Suppose that an extensive form game(a game tree) with finite number of nodes and edges is given. We focus only on the class of games with perfect recall. Let U_i be the set of information sets of player i. For each information set $u \in U_i$, let σ_u be the sequence of actions chosen by i to reach u. Here, a sequence of actions represents the path from the root to a particular node or a leaf. Let C_u be the set of actions at u, and define S_i to be the collection of such sequences for player i.

$$S_i = \{\emptyset\} \cup \{\sigma_u c | u \in U_i, c \in C_u\} \text{ and}$$
$$D_i = \cup_{u \in U_i} C_u$$

A behavior strategy β_i is a map from D_i to $\mathbb{R}$ which satisfies

$$(1)\ \beta_i(c) \geq 0,\ \forall c \in D_i \text{ and}$$
$$(2)\ \sum_{c \in C_u} \beta_i(c) = 1,\ \forall u \in U_i$$

For a sequence s_i of S_i, a realization plan x_i of β_i can be defined as follows

$$x_i(s_i) = \prod_{c \in s_i} \beta_i(c)$$

A typical realization plan x_i should satisfy the following two equations (1) and (2):

$$x_i(\emptyset) = 1 \text{ and} \tag{1}$$
$$-x_i(\sigma_u) + \sum_{c \in C_u} x_i(\sigma_u c) = 0. \tag{2}$$

Here, $x_i(\sigma_u)$ represents the probability of reaching σ_u under x_i, where u is an information set of player i. As we consider only two player games, let x (resp. y) denote the realization plan of player 1 (resp. 2). In matrix notation, constraints (1) and (2) are

$$Ex = e \text{ and}$$
$$Fy = f$$

where e is a vector of $(1, 0, 0, \cdots, 0)^T$ with appropriate dimension, and each row of E represent the coefficients of equation (1). F and f are defined similarly. To demonstrate this complication, consider the following numerical example of figure 2. In this example, the set of

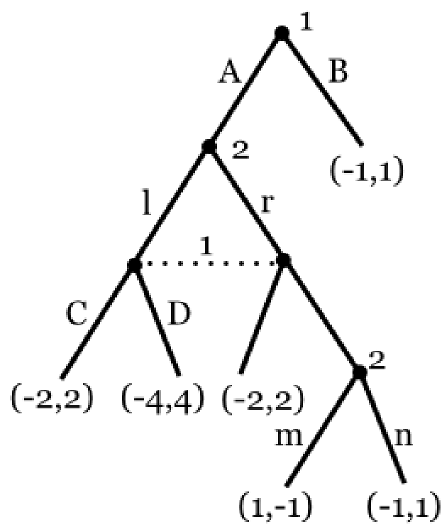

Figure 2: Example 1

sequences are $S_1 = \{\emptyset, A, B, AC, AD\}$ and $S_2 = \{\emptyset, l, r, rm, rn\}$. Corresponding constraints are

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix}, \; F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad e = f = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Now, let A be the payoff matrices of player 1. The row corresponds to S_1 and the column corresponds to S_2 so that the dimension of A is $|S_1| \times |S_2|$. The ij-element of A represents the payoff of player 1 when pair of i-th element of S_1 and j-th element of S_2 is played. The entry is filled with zero if the pair of sequences does not lead to a leaf of the tree. As a result, the payoff has a bilinear structure: if A is the payoff matrix of player 1, $x^T Ay$ is the payoff he expects when a pair of realization plan (x, y) is played, and it is linear in x. Similarly, if we let B be the payoff matrix of the second player, $x^T By$ is linear in y. In this example, the payoff matrix for player 1 is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & -4 & 0 & 1 & -1 \end{pmatrix}.$$

In this sequence form representation, x is a best response to y if and only if it is a solution to

$$\begin{aligned} max_x \quad & x^T A y \\ s.t. \quad & Ex = e \\ & x \geq 0. \end{aligned} \tag{BR1}$$

The program for the second player with his payoff matrix B can be similarly defined. The dual programs are

$$\begin{aligned} min_p \quad & e^T p \\ s.t. \quad & E^T p \geq Ay \end{aligned} \qquad\qquad \begin{aligned} min_q \quad & f^T q \\ s.t. \quad & q^T F \geq x^T B \end{aligned} \tag{N1 and N2}$$

By strong duality, a sufficient and necessity condition for (x, y) being a solution is

$$\begin{aligned} x^T(-Ay + E^T p) = 0 \\ y^T(-B^T x + F^T q) = 0 \end{aligned}$$

with the constraints of the primal and dual programs. In zero-sum cases, since $B = -A$, the program can be further simplified, and a Nash equilibrium of the game can be specified by a solution to the following linear programs

$$\begin{aligned} min_{y,p} \quad & e^T p \\ s.t \quad & E^T p \geq Ay \\ & Fy = f \\ & y \geq 0 \end{aligned} \qquad\qquad \begin{aligned} max_{x,q} \quad & -q^T f \\ s.t. \quad & q^T F \geq x^T(-A) \\ & Ex = e \\ & x \geq 0 \end{aligned}$$

It is easy to see that they are in primal-dual relation.

Remark 1 (von Stengel (1996))**.** The equilibria of a zero-sum game in extensive form with perfect recall are the optimal primal and dual solutions of a linear program whose size, in sparse representation, is linear in the size of the game tree.

Proof. See von Stengel (1996). □

As **Remark 1** says the finding Nash equilibrium in extensive form game is equivalent to solving a linear program, we confirm the fact that Nash equilibrium in extensive form game can be computed in polynomial time.

3.2 ϵ-perturbed game

The bilinearity of payoff and linearity of constraints in sequence-form games make the programs easy to handle. However, it is inadequate to use sequence form to compute an equilibrium that requires local optimality conditions[4] because the realization plan itself cannot describe the choices in the information set that can be reached with zero probability. If a player has a realization plan that reaches one of his information set with zero probability, his subsequent choices after the information set become irrelevant.[5]. One may add an ϵ-perturbation constraint to the program of the following form and try to find its limit equilibrium.

$$\begin{array}{lll} max_x & x^T Ay \\ s.t. & Ex = e \\ & x \geq \epsilon_x^k \end{array} \qquad\qquad \begin{array}{lll} max_y & x^T Ay \\ s.t. & Fy = f \\ & y \geq \epsilon_y^k \end{array}$$

This approach is introduced in Miltersen and Sørensen (2010), who demonstrated how the limit equilibrium of this type represtents a quasi-perfect equilibrium that does not have to be an extensive-form perfect equilibrium due to the bilinearity of the payoff structure. In sequence form, once the player's opponent's realization plan is specified, the player's decision problem becomes a static problem. Before describing this difference in more depth, we recall the definition of extensive form perfect equilibrium.

Definition 1 (Extensive form Perfect Equilibrium)**.** (Selten (1975)) The behavioral strategy profile β of an extensive form game Γ is an extensive form perfect equilibrium if there exists a sequence of perturbed games $\Gamma(\epsilon)$ with $\Gamma(\epsilon) \to \Gamma$ as $\epsilon \to 0$ and β is a limit equilibrium point of $\Gamma(\epsilon)$.

While normal form perfection requires every strategy being played with strictly positive probability in its perturbed game, extensive form perfection uses a perturbed game with every behavior strategy being played with strictly positive probability. Now, see the following Example 2 in Figure 3. The only extensive form perfect equilibrium strategy for the first player is to play L at his first information set, and play r if the second information set if

[4]e.g., extensive form perfect equilibrium
[5]Constraint (2) implies "if $x_i(\sigma_u) = 0$, then $x_i(\sigma_u c) = 0$, $\forall c \in C_u$"

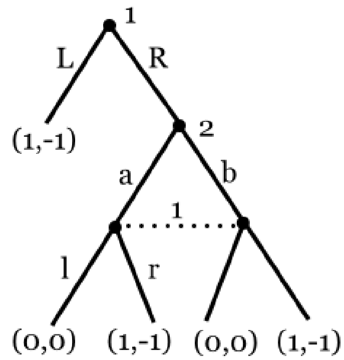

Figure 3: Example 2

reached. Normal form perfect equilibrium, on the other hand, cannot rule out playing R with probability strictly positive probability[6].

Remark 2. The above program with ϵ-perturbation $\epsilon^k = (\epsilon^k_x, \epsilon^k_y) > 0$ cannot solve for an extensive form perfect equilibrium. For any $\epsilon^k \to 0$, there exists a limit of ϵ-equilibrium which is not extensive form perfect.

Proof. Let an arbitrary tremble $\{\epsilon^k\}_{k=1}^{\infty}$ with $\epsilon^k \to 0$ is given. Given $S_1 = \{\emptyset, L, R, Rl, Rr\}$ and $S_2 = \{\emptyset, a, b\}$, we will show that there exists a pair of realization plan converges to $(1,0,1,0,1), (1,1,0)$, which is equivalent to playing $((R,r),(a))$. Let $x = (1, L, R, l, r)$ and $y = (1, a, b)$. As long as $y \geq \epsilon^k_y$, the second player's choice is irrelevant. Thus, simply let $y = (1, 1-\epsilon^k_b, \epsilon^k_b)$. The first player's choice problem is

$$\begin{aligned} max_x \quad & L + r \\ s.t. \quad & L + R = 1 \\ & l + r = R \\ & L \geq \epsilon^k_L,\ R \geq \epsilon^k_R \\ & l \geq \epsilon^k_l,\ r \geq \epsilon^k_r \end{aligned}$$

which is equivalent to

$$\begin{aligned} max_x \quad & 1 - l \\ s.t. \quad & l \geq \epsilon^k_l. \end{aligned}$$

Any combination of realization plans with $l = \epsilon^k_l$ can be an ϵ^k-equilibrium. Thus, $((1,0,1,0,1), (1,1,0))$ is a limit equilibrium of $((1, \epsilon_L, 1-\epsilon_L, \epsilon_l, 1-\epsilon_l), (1, 1-\epsilon_b, \epsilon_b))$. □

[6]Normal form perfect equilibria in pure strategies are $((Ll),(a)), ((Ll),(b)), ((Lr),(a)), ((Lr),(b)), ((Rr),(a)), ((Rr),(b))$.

As we can see from the example, sequence form and its corresponding linear program are not suitable for computing extensive form equilibria: given the other player's realization plan, player 1 chooses her optimal action among $\{L, Rl, Rr\}$ as in normal form games.

In this reason, we will now construct an optimization problem based on behavior strategies. A behavior strategy of i is a map $\beta_i : D_i \to \mathbb{R}$ such that $\beta_i(c) \geq 0$, $\forall c \in D_i$ and $\sum_{c \in C_u} \beta_i(c) = 1$, $\forall u \in U_i$. For a given realization plan x of i, it can be defined by

$$\beta_i(c) = \frac{x(\sigma_u c)}{x(\sigma_u)},\ \forall c \in C_u,\ \forall u \in U_i.$$

Conversely, a realization plan can be inductively recovered from a behavior strategy by

$$x_i(s_i) = \prod_{c \in s_i} \beta_i(c)$$

with $\beta_i(\emptyset) = 1, \forall i$. Now, consider the following program of player 1 based on his behavior strategies.

$$\begin{aligned} max_{\{\beta_1(L),\beta_1(R),\beta_1(l),\beta_1(r)\}} \quad & \beta_1(L) + \beta_1(r)\beta_1(R) \\ s.t. \quad & \beta_1(L) + \beta_1(R) = 1 \\ & \beta_1(l) + \beta_1(r) = 1 \\ & \beta_1(L) \geq \epsilon,\ \beta_1(R) \geq \epsilon \\ & \beta_1(l) \geq \epsilon,\ \beta_1(r) \geq \epsilon \end{aligned}$$

Here, $\beta_1(L) = L,\ \beta_1(R) = R,\ \beta_1(l) = \frac{l}{R},\ \beta_1(r) = \frac{r}{R}$ and $\epsilon > 0$. This non-linear program solves an optimal behavior strategy of player 1, and it is easy to see that the only solution is $(\beta_1(L), \beta_1(R), \beta_1(l), \beta_1(r)) = (1-\epsilon, \epsilon, \epsilon, 1-\epsilon)$.[7] Thus (L, r) is the only limit equilibrium strategy of player 1, and, by definition, it is extensive form perfect. As we can see from the example, a program using behavior strategy as a control variable is usually non-linear. Thus, we will use realization plan, instead of behavior strategy, to ensure linearity. In every ϵ-perturbed game, the two strategic descriptions, behavior strategy and realization plan, are

[7] It is because $\beta_1(r) < 1$ and $\beta_1(L) + \beta_1(R) = 1$, $\forall \epsilon > 0$.

equivalent. In terms of realization plans $((1, L, R, l, r), (1, a, b))$, the program is equivalent to

$$\begin{aligned} max_{L,R,l,r} \quad & L + \frac{r}{R}R \\ s.t. \quad & L + R = 1, \ \frac{l}{R} + \frac{r}{R} = 1 \\ & L \geq \epsilon, \ R \geq \epsilon, \ \frac{l}{R} \geq \epsilon, \ \frac{r}{R} \geq \epsilon \end{aligned}$$

which is also equivalent to

$$\begin{aligned} max_{L,R,l,r} \quad & L + r \\ s.t. \quad & L + R = 1, \ l + r = R \\ & -\epsilon + L \geq 0, \ \ -\epsilon + R \geq 0 \\ & -\epsilon R + l \geq 0, \ \ -\epsilon R + r \geq 0. \end{aligned}$$

Unlike the nonlinear program above, this program is a linear which is efficiently solvable. In this program, higher weight on R necessarily brings higher weight of l, and this fact leads us to the same set of optimal solution.

4 A modified sequence form

To generalize this idea discussed in previous section, consider a general extensive-form games with perfect recall, Γ. Let $\Gamma(\epsilon)$ be a perturbed game of Γ by Selten tremble which assigns $\epsilon \in (0, 1)$ to be the lower bound of each behavior strategy in Γ. In ϵ-perturbed game, behavior strategy $\beta_i(c)$ at information set $u \in U_i$ is constrained by

$$\sum_{c \in C_u} \beta_i(c) = 1 \ \wedge \ \beta_i(c) \geq \epsilon, \ \forall c \in C_u.$$

Here, we impose the following assumption on ϵ which is natural but powerful in our analysis.

$$\epsilon < \max_{u \in U_1 \cup U_2} \frac{1}{|C_u|} \tag{3}$$

where $|A|$ denotes the cardinality of a set A. Thus, we have $\epsilon|C_u| < 1$ for all $u \in U_1 \cup U_2$. As $\beta_i(a) = \frac{x(\sigma_u a)}{x(\sigma_u)}$ and $\beta_i(\emptyset) = 1$, the first constraint is the same as the constraint (1) and (2). The second constraint using realization plan is

$$-\epsilon x(\sigma_u) + x(\sigma_u c) \geq 0. \tag{4}$$

To express this constraint in matrix form, we add a trivial inequality

$$x(\emptyset) \geq 0. \tag{5}$$

We abbreviate constraint (4) and (5) by tremble matrix η_1 of player 1.

$$\eta_1(\epsilon)x \geq 0 \tag{6}$$

The first row of (6) corresponds to inequality (5) and n^{th} row of (6) corresponds to inequality (4) when $\sigma_u c$ is the n^{th} element of S_1. A desired property of $\eta_i(\epsilon)$ is its non-singularity and tractable structure which will not hinder efficient computation of equilibrium. The following **Claim 1** and **Claim 2** show the structure of $\eta_i(\epsilon)$ and $\eta_i^{-1}(\epsilon)$.

Claim 1. For any $\epsilon > 0$ and $i = 1, 2$, $\eta_i(\epsilon)$ is non-singular.

Proof. Note first that we have ones on the diagonal ray. It is because n^{th} row of $\eta_i(\epsilon)$ represents inequality (4) when $\sigma_u c$ is the n^{th} element of S_i, and its coefficient in (4) is one. Moreover, by the definition of a sequence, σ_u always lies before $\sigma_u c$ in S_i for any $u \in U_i$ and $c \in u$. Thus, $\eta_i(\epsilon)$ is an $|S_i| \times |S_i|$ lower triangular matrix with ones on the diagonal. Since the determinant of triangular matrix is equal to the product of all the diagonal element of it, it is non-zero. □

Claim 2. $\eta_i^{-1}(\epsilon)$ is a lower triangular matrix whose elements are either 0 or a polynomial function of ϵ whose value is strictly positive.

Proof. We begin by recalling the following block matrix inversion formula:

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix}.$$

Suppose that S_i, $\epsilon > 0$ and its corresponding $\eta_i(\epsilon)$ are given. We will backward inductively construct $\eta_i^{-1}(\epsilon)$ from the block square matrix starting from the last information set of i.
Base step: The block square matrix is

$$B_1 = \begin{pmatrix} 1 & 0 \\ E_1 & 1 \end{pmatrix}$$

where E_1 is either $-\epsilon$ or 0. The inverse of this matrix is

$$B_1^{-1} = \begin{pmatrix} 1 & 0 \\ -E_1 & 1 \end{pmatrix}.$$

Induction step: Given B_{n-1} and B_{n-1}^{-1}, we have

$$B_n = \begin{pmatrix} 1 & 0 \\ E_n & B_{n-1} \end{pmatrix}.$$

where E_n is either $(-\epsilon, \ldots, -\epsilon, 0, \ldots, 0)^T$ or 0. Using the block matrix inversion formula, the inverse is

$$B_n^{-1} = \begin{pmatrix} 1 & 0 \\ -B_{n-1}^{-1}E_n & B_{n-1}^{-1} \end{pmatrix},$$

and we terminate when $B_n = \eta_i(\epsilon)$. Note that the only calculations required to compute $\eta_i^{-1}(\epsilon)$ are addition and multiplication. Thus, each element in $\eta_i^{-1}(\epsilon)$ is either 0 or a polynomial function of ϵ. $\square$

In our Example 2, we have

$$\eta_1(\epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\epsilon & 1 & 0 & 0 & 0 \\ -\epsilon & 0 & 1 & 0 & 0 \\ 0 & 0 & -\epsilon & 1 & 0 \\ 0 & 0 & -\epsilon & 0 & 1 \end{pmatrix} \quad and \quad \eta_1^{-1}(\epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \epsilon & 1 & 0 & 0 & 0 \\ \epsilon & 0 & 1 & 0 & 0 \\ \epsilon^2 & 0 & \epsilon & 1 & 0 \\ \epsilon^2 & 0 & \epsilon & 0 & 1 \end{pmatrix}.$$

We define a modified sequence form of the original game Γ and Selten tremble ϵ by $\Gamma(\epsilon)$. Using these parametric tremble matrices $(\eta_1(\epsilon), \eta_2(\epsilon))$, player 1's problem to find a best response to y in our ϵ-perturbed game is

$$\begin{aligned} max_x \quad & x^T Ay \\ s.t. \quad & Ex = e \\ & \eta_1(\epsilon)x \geq 0. \end{aligned} \tag{P1}$$

The dual linear program of $(P1)$ is

$$\begin{aligned} min_p \quad & e^T p \\ s.t. \quad & \eta_1^{-1}(\epsilon)^T[E^T p - Ay] \geq 0. \end{aligned} \tag{D1}$$

Here, $\eta_i^{-1}(\epsilon)^T$ refers to the transpose of $\eta_i^{-1}(\epsilon)$. By strong duality, the solutions are optimal *if and only if* the duality gap is zero. That is

$$x^T(E^T p - Ay) = 0.$$

Similarly, we can construct primal program (P2) and its dual program (D2) for player 2 as follows

$$
\begin{array}{llll}
max_y & x^T B y & \qquad min_q & -q^T f \\
s.t. & Fy = f & \qquad s.t. & \eta_2^{-1}(\epsilon)^T [F^T q - B^T x] \geq 0 \\
 & \eta_2(\epsilon) y \quad \geq 0 & &
\end{array}
$$

Thus, an equilibrium of $\Gamma(\epsilon)$ can be found from the following program.

Theorem 1. An equilibrium of $\Gamma(\epsilon)$ is solution to the following linear complementarity program:

$$
\begin{aligned}
x^T(E^T p - Ay) &= 0 \qquad \text{(LCP}(\epsilon))\\
y^T(F^T q - Bx) &= 0 \\
Ex = e,\ Fy &= f \\
\eta_1(\epsilon)x \geq 0,\ \eta_2(\epsilon)y &\geq 0 \\
\eta_1^{-1}(\epsilon)^T[E^T p - Ay] &\geq 0 \\
\eta_2^{-1}(\epsilon)^T[F^T q - B^T x] &\geq 0.
\end{aligned}
$$

Proof. The complementary pairs here are $(x^T\eta_1(\epsilon)^T, \eta_1^{-1}(\epsilon)^T[E^T p - Ay])$ and $(y^T\eta_2(\epsilon)^T, \eta_2^{-1}(\epsilon)^T[F^T q - B^T x])$. It is sufficient to show that $(P1)$ is a dual program of $(D1)$. The dual program of $(P1)$ is

$$
\begin{array}{ll}
max_{p,\mu} & \phi(p,\mu) \\
s.t. & \mu \geq 0
\end{array}
$$

where

$$
\begin{aligned}
\phi(p,\mu) &= \inf_x \Big[- x^T A y + p^T[Ex - e] - \mu^T \eta_1(\epsilon) x \Big] \\
&= \inf_{p,y} \Big[x^T[-Ay + E^T p - \eta_1(\epsilon)^T \mu] - p^T e \Big]
\end{aligned}
$$

Infimum of the functional value exists only if $-Ay + E^T p = \eta_1(\epsilon)^T \mu$. Since $\eta_1(\epsilon)$ is non-singular, the dual program is $(D1)$. Similarly, $(D2)$ is a dual program of $(P2)$. □

When $\epsilon = 0$, the program collapses to the program of Koller, Megiddo, and von Stengel (1996) which finds a Nash equilibrium of Γ. Let $(x(\epsilon), y(\epsilon))$ be a solution to the above linear complementarity program. An immediate corollary follows from the maximum theorem of Berge[8].

[8]Let X and A be metric spaces, $f : X \times A \to \mathbb{R}$ be a function jointly continuous in its two arguments,

Corollary 1. In two-player games with perfect recall, a Nash equilibrium exists and an extensive form perfect equilibrium is a Nash equilibrium.

Proof. We can add a vacuous constraint $\eta_1^{-1}(\epsilon)^T[E^Tp - Ay] \leq (1, \ldots, 1)^T$ to $(D1)$ and $\eta_2^{-1}(\epsilon)^T[F^Tq - B^Tx] \leq (1, \ldots, 1)^T$ to $(D2)$. By the maximum theorem of Berge, each best response mapping is upper semicontinuous at $\epsilon = 0$. Thus, any limit equilibrium point should be an equilibrium point when $\epsilon = 0$. □

5 Computation of extensive-form perfect equilibrium

As we can see from **Corollary 1**, the optimal solution mapping is upper semi-continuous by the Maximum theorem of Berge, but it does not have to be lower semi-continuous at $\epsilon = 0$. For example, the game in Example 2 has an equilibrium point of $((R, r), a)$ at $\epsilon = 0$, but there does not exists a sequence of ϵ-equilibrium tending toward it. In order to find a limit equilibrium of $\{\Gamma(\epsilon)\}_{\epsilon \to 0}$, we need to find a lower semi-continuous path of the mapping. The result to be presented in this paper is the following.

Theorem 2. For a sufficiently small $\epsilon > 0$, an extensive form perfect equilibrium (x, y) can be recovered from (x^ϵ, y^ϵ), a solution to $\Gamma(\epsilon)$.

Proof. Suppose that $\epsilon > 0$ and its corresponding equilibrium (x^ϵ, y^ϵ) of $\Gamma(\epsilon)$ are given. We inductively recover x by the following formula starting from $\sigma_u = \emptyset$.

$$x(\sigma_u c) = \frac{1}{1 - |C_u|\epsilon}\left[\frac{x^\epsilon(\sigma_u c)}{x^\epsilon(\sigma_u)} - \epsilon\right]x(\sigma_u). \tag{7}$$

Note first that, for each $c \in u$, we have

$$\begin{aligned}
\sum_{c \in u} x(\sigma_u c) =& \sum_{c \in u} \frac{x(\sigma_u)}{1 - |C_u|\epsilon}\left[\frac{x^\epsilon(\sigma_u c)}{x^\epsilon(\sigma_u)} - \epsilon\right] \\
=& \frac{x(\sigma_u)}{1 - |C_u|\epsilon}\sum_{c \in u}\left[\frac{x^\epsilon(\sigma_u c)}{x^\epsilon(\sigma_u)} - \epsilon\right] \\
=& x(\sigma_u)\left[\frac{1}{1 - |C_u|\epsilon} - \frac{|C_u|\epsilon}{1 - |C_u|\epsilon}\right] = x(\sigma_u).
\end{aligned}$$

Moreover, for any $c \in u$ and $u \in U_1$, $x(\sigma_u c) \geq 0$ follows from $\eta_1(\epsilon)x^\epsilon \geq 0$ and $|C_u|\epsilon < 1$. Thus, x is a realization plan and as ϵ vanishes, x^ϵ converges to x. Now, we will show that there's a sequence of realization plan $\{(x^{\epsilon^k}, y^{\epsilon^k})\}_{k=1}^\infty$ where each is a solution to $\Gamma(\epsilon^k)$ and $(x^{\epsilon^k}, y^{\epsilon^k}) \to (x, y)$. For any $\epsilon' \in (0, \epsilon)$, define $x^{\epsilon'}$ in a similar way: 1) $x^{\epsilon'}(\sigma_u c) = \epsilon'$ *if and*

and $g : A \rightrightarrows X$ be a compact-valued correspondence. For $x \in X$ and $a \in A$, let $\hat{f}(a) = \max_{x \in g(a)} f(x, a)$ and $\hat{g}(a) = \arg\max_{x \in g(a)} f(x, a)$. If g is continuous at a', then $\hat{f}$ is continuous at a' and $\hat{g}$ is non-empty, compact-valued, and upper semi-continuous at a'.

only if $x^\epsilon(\sigma_u c) = \epsilon$ and 2) For $u \in U$ and $c_1, c_2 \in u$, if $x^\epsilon(\sigma_u c_1) > \epsilon$ and $x^\epsilon(\sigma_u c_2) > \epsilon$ then $\frac{x^{\epsilon'}(\sigma_u c_1)}{x^{\epsilon'}(\sigma_u c_2)} = \frac{x^\epsilon(\sigma_u c_1)}{x^\epsilon(\sigma_u c_2)}$ comply with the constraint $\sum_{c \in u} x^{\epsilon'}(\sigma_u c) = x^{\epsilon'}(\sigma_u)$. If there is only one $c \in u$ such that $x^\epsilon(\sigma_u c) > \epsilon$, then we simply set $x^{\epsilon'}(\sigma_u c) = 1 - |C_u - 1|\epsilon$. In this way, for any sequence $\epsilon^k \to 0$ with $\epsilon^k < \epsilon$, we have $x^{\epsilon^k} \to x$. We can similarly construct y^{ϵ^k} from y^ϵ. Now, we will show that $x^{\epsilon'}$ is a best response to $y^{\epsilon'}$ in $\Gamma(\epsilon')$. This will suffice to show (x, y) is the limit equilibrium, and hence an extensive form perfect equilibrium.

Lemma 1. $y(\epsilon)$ induces a linear preference over choices at each information set u of player 1.

Proof. tba □

Lemma 2. For any $\zeta_x > 0$, $\exists \epsilon_{\zeta_x} > 0$ s.t. difference between slopes of indifference curves induced by $y(\epsilon')$ and by $y(\epsilon)$ is bounded by ζ_x, $\forall \epsilon' \in (0, \epsilon)$, $\forall u \in U$.

Proof. tba □

By **Lemma 1** and **Lemma 2**, we can set ζ_x small so that optimal choices of player 1 at each information set do not change over any $\epsilon' \leq \epsilon_{\zeta_x}$. This means not only $x^{\epsilon_{\zeta_x}}$, but also $x^{\epsilon'}$ is a best response to $y^{\epsilon'}$, $\forall \epsilon' \leq \epsilon_{\zeta_x}$. Similarly, we can find ζ_y and ϵ_{ζ_y} which does not change optimal choices of player 2. Thus, if we set ϵ^k by $\frac{1}{k}\min\{\epsilon_{\zeta_x}, \epsilon_{\zeta_y}\}$, then (x, y) is a limit equilibrium point of $(x^{\epsilon^k}, y^{\epsilon^k})$ and $\Gamma(\epsilon^k) \to \Gamma(0)$. □

The construction of x is based on a well-know fact that a best response to opponents' strategy is a face of a simplex. In our program, each $Ay(\epsilon)$ induces a linear preference over C_u for all $u \in U_1$. Thus, if a sequence σ leading to a terminal node is preferred to any other sequence in $\Gamma(\epsilon)$, it is true that σ is still preferred in program without no ϵ restriction on x if ϵ is sufficiently small.

In zero-sum cases, we have the standard form of finding equilibrium with $B = -A$.

Claim 3. An equilibrium of $\Gamma(\epsilon)$ is a solution to the following programs $ZS_1(\epsilon)$ and $ZS_2(\epsilon)$:

$$\begin{aligned} min_{p,y} \quad & p^T e \\ s.t. \quad & \eta_1^{-1}(\epsilon)^T [E^T p - Ay] \geq 0 \\ & Fy = f \\ & \eta_2(\epsilon) y \geq 0 \end{aligned} \qquad\qquad \begin{aligned} min_{q,x} \quad & -q^T f \\ s.t. \quad & \eta_2^{-1}(\epsilon)^T [A^T x - F^T q] \geq 0 \\ & Ex = e \\ & \eta_1(\epsilon) x \geq 0. \end{aligned}$$

Proof. Since $ZS_1(\epsilon)$ solves (D1) and $ZS_2(\epsilon)$ solves (D2), it suffices to show that they are in primal-dual relation. The dual of $ZS_1(\epsilon)$ is

$$\begin{aligned} max_{\lambda,\mu_1,\mu_2} \quad & \phi(\lambda, \mu_1, \mu_2) \\ s.t. \quad & \mu_1 \geq 0,\ \mu_2 \geq 0 \end{aligned}$$

where

$$\begin{aligned} \phi(\lambda, \mu_1, \mu_2) &= \inf_{p,y} \Big[p^T e + \lambda^T \big[f - Fy \big] - \mu_1^T \eta_1^{-1}(\epsilon)^T \big[E^T p - Ay \big] - \mu_2^T \big[\eta_2(\epsilon)^T y \big] \Big] \\ &= \inf_{p,y} \Big[\big[e^T - \mu_1^T \eta_1^{-1}(\epsilon)^T E^T \big] p + \big[-\lambda^T F + \mu_1^T \eta_1^{-1}(\epsilon)^T A - \mu_2^T \eta_2(\epsilon)^T \big] y + \lambda^T f \Big] \end{aligned}$$

Infimum exists only if $e - \mu_1^T \eta_1^{-1}(\epsilon)^T E^T = 0$ and $-\lambda^T F + \mu_1^T \eta_1^{-1}(\epsilon)^T A - \mu_2^T \eta_2(\epsilon)^T = 0$. Thus, the dual program is

$$\begin{aligned} max_{\lambda,\mu_1,\mu_2} \quad & \lambda^T f \\ s.t. \quad & \mu_1 \geq 0,\ \mu_2 \geq 0 \\ & \mu_1^T \eta_1^{-1}(\epsilon)^T E^T = e \\ & -\lambda^T F + \mu_1^T \eta_1^{-1}(\epsilon)^T A - \mu_2^T \eta_2(\epsilon)^T = 0. \end{aligned}$$

Here, let $\lambda = q$ and $\eta_1^{-1}(\epsilon)\mu_1 = x$. Since $-F^T\lambda + A^T \eta_1^{-1}(\epsilon)\mu_1 - \eta_2(\epsilon)\mu_2 = 0 \Leftrightarrow \eta_2^{-1}(\epsilon)[-F^T\lambda + A^T \eta_1^{-1}(\epsilon)\mu_1] = \mu_2$, we have the dual program $ZS_2(\epsilon)$. □

As a result of **Theorem 2** and **Claim 3**, the following procedure will compute an extensive form perfect equilibrium in zero-sum games and general-sum games. 1) Find and fix an $\epsilon > 0$, 2) Given ϵ, solve LCP(ϵ) in general-sum case (or $ZS_1(\epsilon)$ in zero-sum case) and 3) Recover (x, y) from (x^ϵ, y^ϵ). In our **Example 2**, let $\epsilon = .1$. The only solution to $ZS_2(.1)$ is $(\emptyset^\epsilon, L^\epsilon, R^\epsilon, Rl^\epsilon, Rr^\epsilon) = (1, .9, .1, .01, .09)$. Applying the formula (7), we get $(1, 1, 0, 0, 0)$ as an extensive form perfect equilibrium.

References

N. Gatti and C. Iuliano. Computing an extensive-form perfect equilibirum in two-player games. Twenty-Fifty AAAI Conference on Artificial Intelligence, 2011.

N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4 (4):373–395, 1984.

L. G. Khachiyan. A polynomial algorithm in linear programming. *Doklady Akademiia Nauk SSSR*, 244:1093–1096, 1979.

L. G. Khachiyan. Polynomial algorithms in linear programming. *USSR Computational Mathematics and Mathematical Physics*, 20(1):53–72, 1980.

D. Koller, N. Megiddo, and B. von Stengel. Efficient computation of equilibria for extensive two-person games. *Games and Economics Behavior*, 14(51):247–259, 1996.

C. E. Lemke and J. J. T. Howson. Equilibrium points of bimatrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12(2):413–423, 1964.

J.-F. Mertens. Two examples of strategic equilibrium. *Games and Economics Behavior*, 8: 378–388, 1995.

P. B. Miltersen and T. B. Sørensen. Computing a quasi-perfect equilibrium of a two-player game. *Economic Theory*, 42(1):175–192, 2010.

R. Selten. Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory*, 4(1):25–55, 1975.

E. van Damme. A relation between perfect equilibria in extensive form games and proper equilibria in normal form games. *International Journal of Game Theory*, 13:1–13, 1984.

E. van Damme. Stability and perfection of nash equilibria. *Berlin: Springer-Verlag*, 1987.

B. von Stengel. Efficient computation of behavior strategies. *Games and Economics Behavior*, 14(0050):220–246, 1996.

B. von Stengel, A. van den Elzen, and D. Talman. Computing normal form perfect equilibria for extensive two-person games. *Econometrica*, 70(2):693–715, 2002.