Silence Speaks Volumes

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Preliminary Draft, Comments Welcome!

Abstract

We consider a sender-receiver game in a continuous time framework, in which the biased sender, who has private information, can send a cheap-talk message anytime to the receiver. The receiver also can choose to stop and make a decision anytime. Time is assumed to be costly. In the study, we characterize a particular class of equilibrium which explains "suspicion grows from silence" phenomenon. In equilibrium, the sender gradually reveals his type, and the receiver learns the type over time. However, a full revelation from all types of the sender is not possible because of the cost of time.

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1. Introduction

The purpose of this study is to explore silence as a means of information transmission. Efficient revelation of information requires both verbal and non-verbal messages. For example, when companies release news, whether it is hard information or soft information, they not only deliberately choose speeches, tones and words, but they also choose where and when to make the announcement. However, there are some relevant questions raised, such as what kind of information can a non-verbal message convey, especially the timing decision on the delivery of the message? Can a firm deliver more detailed information by choosing when to release news? How much of the details can be transmitted if it is possible?

In this study, I focus on a simple but suggestive model of a sender-receiver game in a continuous time framework, in which the biased sender, who has private information, can choose when to send a cheap-talk message to a receiver, and the receiver can also choose when to stop waiting and make a decision. Time is assumed to be costly in the sense that players discount future payoffs. In this setting, I show that there exists an equilibrium in which the sender uses timing of the message as a revelation device if the bias of the sender is not so large. In the equilibrium, the sender gradually reveals its type and the receiver gradually learns the sender's type up to a certain point in time. However, full revelation of information is not possible from all types of the senders because of the cost of waiting. If the receiver believes that most of the information is released, then he is better off stopping and making a guess rather than waiting to learn more about the sender's type.

Three remarks are in order. First, the option to keep silent dramatically increases the quality of information that can be transmitted in equilibrium. Choosing how much to keep silent before delivering a message works as a signaling device to convey information. However, because waiting is costly, there is a limitation in the amount of information that can be conveyed using this signaling device. If the cost incurred to the receiver due to the waiting outweighs the benefit from extra information the receiver infers from the silence, he stops waiting and makes a decision under uncertainty. Second, in the equilibrium in consideration of this study, delay in decision making is a part of equilibrium behavior. This is because the revelation and learning of the sender's information evolves gradually over time, waiting and delay in decision making is a part of equilibrium behavior. Last, the equilibrium belief in the model supports the "suspicion grows from silence" phenomenon. As silence from the sender gets longer, the receiver's belief on that sender is of certain type gets stronger. What is more is that the direction of the formation of suspicion in equilibrium agrees with the direction of bias in the model.

2. Related Literature

There is a body of literature that introduces prolonged communication protocol to the original model of Crawford and Sobel (1982), and obtains an improvement in the quality of information transmitted from a sender (Aumann and Hart (2003), Krishna and Morgan (2004) and Golosov et al. (2014)). Our paper belongs to this strand of cheap talk literature. Golosov et al. (2014) considers finitely repeated cheap talk game, and it is shown that full-revelation from a biased sender is possible under some model parameters. In their model, the state of the world is held fixed over time and payoff realizations are observable at the end of each period. A trigger strategy is used to sustain the fully-revealing equilibrium. In Renault et al. (2013), the infinitely repeated cheap talk game is considered, and trigger strategy of the sender is also used to sustain the set of equilibrium payoffs. However, our paper fundamentally differs from such papers because payoffs in our paper are realized only once when the decision is made. (In this reason, punishment based on the other player's past action is not available.) A full revelation of information is also possible when there are multiple number of senders (Battaglini (2002) and Esö and Fong (2010)). Esö and Fong (2010) is closely related to our paper as it also studies dynamic cheap talk game with costly delay. In their model, there are multiple senders, and a fully revealing equilibrium exists if 1) senders commonly know the underlying state and 2) they simultaneously send messages to the receiver. The receiver in their model cross-checks the senders' messages to learn the true state. In their equilibrium, delay does not occur, which differs from our paper's main result. Lastly, in cheap talk literature, our paper is related to cheap talk with costly signal. In Austen-Smith and Banks (2000) and Kartik (2007), the sender can send a costly signal along with a costless cheap talk, and they show that a more informative message can be sent in equilibrium than any equilibrium in Crawford and Sobel (1982). Although our model allows costly actions to be played by any party, our paper shares some similarity with that paper: in both papers, social loss occurs in equilibrium.

Another strand of literature that is closely related to our paper is a literature on optimal stopping problem. Kruse and Strack (2015) studies optimal stopping problem when one side is endowed with private information. In the paper, the principal-agent problem is considered, and the the principal can commit to a monetary transfer to influence the agent's stopping decision. Grenadier et al. (2015) considers an optimal timing to exercise a real option when the option holder is advised by a biased expert. The decision maker has to choose when to exercise an option, and if he is advised by an expert who is biased towards late exercise, then it is shown that a fully informative equilibrium with delay exists. However, our study is different from that paper as the decision maker in our paper chooses not only when to stop listening to the expert, but also what to choose when he chooses to stop.

3. Model

3.1. Basic setup

Time and Type.– Time is continuous and the horizon is infinite, indexed by $t \in [0, \infty)$. At time 0, a value θ is realized from a random variable Θ, whose support is $[0, 1]$, with CDF F and pdf f.

Players and Asymmetric information.– There are two players, an informed party (sender) and an uninformed party (receiver). The realized value θ is payoff relevant to the both players, but it is only privately observed by the sender. There is a single decision to be made, and the receiver is the one who makes the decision. Therefore, the sender may want to communicate with the receiver to influence his decision. Communication is assumed to be costless and unverifiable.

Heuristically, the timing of events over an infinitesimal size of interval $[t, t + dt)$ can be described as follows:

0. True state θ is realized at $t = 0$ and the receiver has not made a non-waiting decision up to time t.

(i) The sender either keeps silent, $\emptyset^S$, or sends a costless and unverifiable message $m_t \in M$ to the receiver.

(ii) The receiver either wait or make a decision. If he choose to wait, $\emptyset^R$, the game continues to the next interval starting at $t + dt$. However, if he makes a decision $a \in A = \Theta$, payoffs are realized and the game ends.

We assume that the message space M contains Θ so that the sender is not bothered by a capacity constraint.

Payoff.– Delay in decision and communication is costly and wasteful in the sense that players discount future payoff: there is a common discount factor $r > 0$. If a player expects to obtain a certain payoff of U at time t away from today, his current expected utility is $e^{-rt}U$. There is a conflict of interest which is measured in a scalar parameter b. If a non-waiting decision a is made when the state is θ, the sender obtains payoff of $u^S(a, \theta, b)$ and the receiver obtains $u^R(a, \theta)$. Otherwise, if the receiver chooses to wait, they receive 0 stage payoff and the game continues.

Both u^R and u^S are non-negative for all possible (a, θ, b) tuple and satisfy the following **Condition CS**.

Assumption 1 (Condition CS)**.** u^i is twice differentiable, $u^i_{11} < 0$ and $u^i_{12} > 0$, for each $i \in \{S, R\}$.

Equilibrium.– Let $h^S_t = \{\theta\} \cup \{\tilde{m}_\tau\}_{\tau<t}$ denote the sender's history at t, where $\tilde{m}_\tau \in M \cup \{\emptyset^S\}$, $\forall\tau$, and let $\mathscr{H}^S$ denote the collection of all possible history of the sender for $t \geq 0$[1]. Similarly, let $h^R_t = \{\tilde{m}_\tau\}_{\tau\leq t}$ denote the receiver's history at t and define $\mathscr{H}^R$ analogously. A strategy of the sender is a map $\sigma : \mathscr{H}^S \to M \cup \{\emptyset^S\}$. The sender keeps silent or chooses a message based on what he has observed and his prior messages he sent to the receiver. A strategy ρ of the receiver is a map $\rho : \mathscr{H}^R \to A \cup \{\emptyset^R\}$. Hence, after receiving messages or observing inactions of the sender, the receiver decides not only whether to stop or not, but also which action to take if he decides to stop. The receiver's belief at t is a map $\mu(\cdot|h^R_t) : \mathscr{B} \to [0, 1]$, where $\mathscr{B}$ is a Borel set defined over the support of F. The receiver's belief system is a collection $\{\mu(\cdot|h)\}_{h\in\mathscr{H}^R}$, which we will denote by μ. The equilibrium concept we employ is perfect Bayesian equilibrium(hereafter, an equilibrium).

Given the receiver's belief system μ, we define a stopping time after history h^R_t by $T(h^R_t, \sigma, \rho) = \inf\{t \geq 0 : \rho(h^R_t \cup \{\sigma(h^S_\tau)\}_{\tau\geq t}) \neq \emptyset^R\}$.[2] In words, T is the time when the receiver stops and make a decision when players abide with strategy profiles (σ, ρ) after some history h^R_t given his belief μ.

Definition 1. A strategy pair (σ^S, σ^R) and a belief system μ constitutes an equilibrium if it satisfies

(i) (Sender's Sequential rationality) For any $t \leq T^*$, h^S_t and any other strategy $\bar{\sigma}$,

$$\mathbb{E}\big[e^{-r(T^*-t)}u^S(\rho(h^R_{T^*}), \theta_0, b)\big|h^S_t, \mu\big] \geq \mathbb{E}\big[e^{-r(\bar{T}-t)}u^S(\rho(\bar{h}^R_{\bar{T}}), \theta_0, b)\big|h^S_t, \mu\big] \quad (1)$$

where $T^* = T(h^R_t, \sigma, \rho)$, $\bar{T} = T(h^R_t, \bar{\sigma}, \rho)$, $h^R_{T^*} = h^R_t \cup \{\sigma(h^S_\tau)\}_{\tau\in[t,T^*]}$ and $\bar{h}^R_{\bar{T}} = h^R_t \cup \{\bar{\sigma}(\bar{h}^S_\tau)\}_{\tau\in[t,\bar{T}]}$.

(ii) (Receiver's Sequential rationality) For any $t \leq T^*$, h^R_t and any $T' \geq t$ and $a' \in A$

$$\mathbb{E}\big[e^{-r(T^*-t)}u^R(\rho(h^R_{T^*}), \theta)\big|h^R_t, \mu\big] \geq \mathbb{E}\big[e^{-r(\bar{T}'-t)}u^R(a', \theta)\big|h^R_t, \mu\big] \quad (2)$$

[1] We leave receiver's history of action out because it up to t consists of nothing but $\emptyset^R$.

[2] Note that $\{h^S_\tau\}_{\tau\geq t}$ is fully pre-described by h^R_t, σ and ρ.

(iii) (Consistency of belief) The receiver updates his belief using Bayes' rule whenever possible. That is,

$$\mu(\theta|h_t^R \cup \{m_t\}) = \frac{\mu(\theta|h_t^R)\mathbf{1}\{\sigma(h_t^S) = m_t\}}{\int_0^1 \mu(\theta'|h_t^R)\mathbf{1}\{\sigma(\theta', h_t^R) = m_t\}d\theta'}. \tag{3}$$

3.2. Uniform - Quadratic Case

We first consider uniform-quadratic setup introduced by Crawford and Sobel (1982): 1) Θ follows a uniform distribution; 2) $u^R = -(\theta - a)^2 + 1$ and 3) $u^S = -(\theta + b - a)^2 + 1$. A scalar 1 is added to make utilities be higher than utility from players' outside option, indefinite waiting. Suppose first the case of positively biased senders, $b > 0$ case.

Theorem 1. For small $|b|$, there exists an equilibrium in which

(i) Sender type $\theta \leq \bar{\theta}$ truthfully reveals his type at a revelation time $\kappa(\theta)$ and

(ii) The other senders plays a partition equilibrium of Crawford and Sobel (1982) at $\kappa(\bar{\theta})$).

Thus, in this equilibrium, receiver gradually learns the true state, and his belief continuously puts more weights on the event that sender's being high type.

Proof. See **Appendix**. ■

When the sender is negatively biased, $b < 0$, the opposite holds true.

Corollary 1. For small $|b|$, there exists an equilibrium in which

(i) Sender type $\theta \geq \underline{\theta}$ truthfully reveals his type at a revelation time $\underline{\kappa}(\underline{\theta})$ and

(ii) The other senders plays a partition equilibrium of Crawford and Sobel (1982) at $\underline{\kappa}(\underline{\theta})$).

The proof is relegated to **Appendix**. The intuition is clear.

Consider first the sender's incentive. Under the true state θ, suppose that the receiver believes whichever the sender tells him[3]: if $\tilde{\theta}$ is received, the receiver believes the true state is $\tilde{\theta}$, and chooses $\tilde{\theta}$ without delay. Given this receiver's behavior, the sender's preference is convex in $(\tilde{\theta}, -t)$-plane, and the slope of indifference curves over the plane is 1) independent of t and 2) the slope at $\tilde{\theta} = \theta$ is a constant κ for all θ. Thus, by making the revelation rule tangent to the indifference curve at θ, the sender expects the highest utility when he tells the

[3] We can assume this as it is the equilibrium behavior.

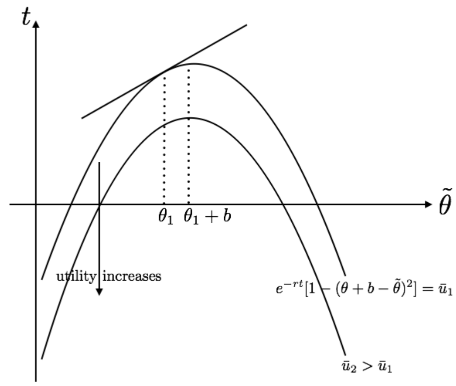

Figure 1: Sender's indifference curves

truth when the true state is θ. Using this property, if the revelation rule is an upper envelope of the sender's indifference curve for all possible $\theta \in \Theta$, no type of sender has an incentive to deviate from truth-telling. This linear revelation rule can be sustained by a simple message strategy of the sender: type θ sender pools with type $\theta' > \theta$ before $t = \kappa\theta$, and reveals his type at time $t = \kappa\theta$.

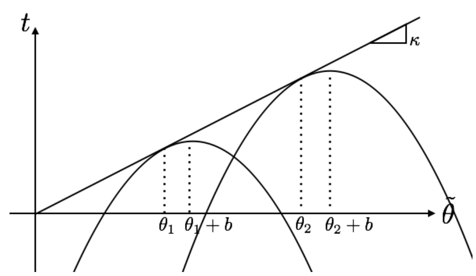

Figure 2: Revelation strategy

Under this strategy of the sender, the receiver faces a dynamic programming problem. At time t, he has to choose whether to stop and take a guess or to continue. Comparing his benefits between the two options, smaller κ makes the second option more profitable. It is because, under the sender's strategy, the smaller κ is, the more information is revealed in a unit of time. In an extreme case with zero-bias($\kappa = 0$), full information is revealed at once as we can see from Figure 4.

Some features worth mentioning arise in the equilibrium. First, waiting is not used to threat the other party. Instead, it occurs because learning takes time in equilibrium. Second, even if the sender tries to reveal his type earlier than the scheduled time, it is not possible. It is because the receiver will consider the sender's message as an out-of-equilibrium message. The only way to prompt the receiver to make an early decision is to pretend to be other type. However, this

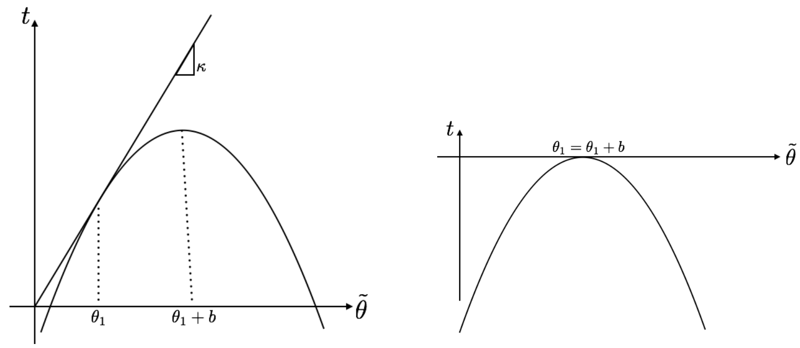

Figure 3: Bias and Revelation strategy

would harm his expected payoff.

Appendix

Proof of Theorem 1

Proof. Let $M = \{m_0, m_1, m_2, \cdots\}$ be any countable set and consider a linear time revelation rule: type $\theta \leq \bar{\theta}$ sender keeps silent until he sends m_0 at $t = \kappa\theta$, where $\kappa = \frac{2b}{r(1-b^2)}$. While $t \leq \kappa\bar{\theta} \equiv T$, the receiver waits for a message m_0, and plays his most preferred action under his belief when the message arrives. Otherwise, if $t = T$ is reached without sending and receiving m_0, both players plays an equilibrium described by Crawford and Sobel (1982) with the receiver's updated information: $f(\theta) = \frac{1}{1-\bar{\theta}}$, $\forall\theta \geq \bar{\theta}$ and zero otherwise. Using Bayes' rule, the receiver's belief evolves as follows. $\mu([\theta, 1]|h_t^R) = 0$ if $t < \kappa\theta$ and h_t^R contains m_0, and $\mu([\theta, 1]|h_t^R) = 1 - F(t/\kappa)$ if it does not contain m_0. For any message $m \notin M$, The receiver does not update his belief as it is out-of-equilibirum message.

Suppose that, at time $t \geq 0$, type $\theta \leq \bar{\theta}$ sender tries to mimic type θ' where $\theta' \neq \theta$ and $\frac{t}{\kappa} \leq \theta' \leq \bar{\theta}$.[4] By doing so, he expects

$$e^{-r(\kappa\theta' - t)}\big[-(\theta + b - \theta')^2 + 1\big]$$

which is maximized at $\theta' = \theta$ if $\kappa = \frac{2b}{r(1-b^2)}$. To see this, the first and the second

[4] Receiver at t knows the sender's type is strictly above $\frac{t}{\kappa}$, and thus he does not have any incentive to mimic any type below $\frac{t}{\kappa}$.

order derivatives with respect to θ' are provided below.

$$-r\kappa e^{-r(\kappa\theta'-t)}[-(\theta+b-\theta')^2+1]+2e^{-r(\kappa\theta'-t)}(\theta+b-\theta') \qquad \text{(1st derivative)}$$
$$r^2\kappa^2 e^{-r(\kappa\theta'-t)}[-(\theta+b-\theta')^2+1]-4r\kappa e^{-r(\kappa\theta'-t)}(\theta+b-\theta')-2e^{-r(\kappa\theta'-t)} \qquad \text{(2nd derivative)}$$

Plugging in $\kappa=\frac{2b}{r(1-b^2)}$ yields,

$$-\frac{2b}{1-b^2}e^{-\frac{2b}{1-b^2}\theta'+rt}[1-(\theta+b-\theta')^2]+2e^{-\frac{2b}{1-b^2}\theta'+rt}(\theta+b-\theta') \qquad \text{(1st derivative)}$$

which is zero at $\theta'=\theta$, and

$$\left[\left(\frac{2b}{1-b^2}\right)^2[1-(\theta+b-\theta')^2]-\frac{8b}{1-b^2}(\theta+b-\theta')-2\right]e^{-\frac{2b}{1-b^2}\theta'+rt} \qquad \text{(2nd derivative)}$$

whose value is $\left[-\frac{8b^2}{1-b^2}-2\right]e^{-\frac{2b}{1-b^2}\theta+rt}<0$. To check incentives not to mimic $\theta'>\bar{\theta}$, note that if he keeps silent by $\kappa\bar{\theta}$, the receiver will choose an action $a'>\bar{\theta}$ at T. His expected utility at time $t<T$ from this is

$$e^{-r(T-t)}(-(\theta+b-a')^2+1)$$

which again is maximized at $a'=\theta$. Now consider a sender with $\theta>\bar{\theta}$ who expects a' to be played in equilibrium. At time t, if he pretends to be $\frac{t}{\kappa}\leq\theta'\leq\bar{\theta}$, his expected payoff is

$$e^{-r(\kappa\theta'-t)}(-(\theta+b-\theta')^2+1).$$

We can show that derivative is positive as long as $\theta\geq\theta'$. Thus, we have

$$\begin{aligned} e^{-r(\kappa\theta'-t)}(-(\theta+b-\theta')^2+1) &\leq e^{-r(\kappa\bar{\theta}-t)}(-(\theta+b-\bar{\theta})^2+1) \\ &\leq e^{-r(\kappa\bar{\theta}-t)}(-(\theta+b-a')^2+1). \end{aligned}$$

The latter inequality is by our choice of $\bar{\theta}$ and a'. Since players play a Crawford and Sobel (1982) cheap talk game over a restricted state space $(\bar{\theta},1]$, a' should be the most preferred action to the sender among all actions arise in equilibrium of the cheap talk game at $t=T$.

Before we discuss the receiver's incentives, we will pin down $\bar{\theta}$ and actions played at the cheap talk phase. Suppose that action a_i is being played in response to message m_i at T, $i \in \{1, \cdots, n\}$[5], and suppose that $a_i < a_j$ whenever $i < j$. $\bar{\theta}$ is decided by "arbitrage" condition

$$-(\bar{\theta} + b - \bar{\theta})^2 + 1 = -(\bar{\theta} + b - a_1)^2 + 1$$

That is, $\bar{\theta}$ type sender is indifferent between telling the truth at time T and his most preferred action among static cheap talk outcome, $\{a_1, \cdots, a_n\}$.[6] Since $a_1 = \frac{\theta_0 + \theta_1}{2}$, it is

$$\bar{\theta} = \theta_1 - 4b.$$

Recall that any equilibrium of Crawford and Sobel (1982) in uniform-quadratic setting can be represented by a system of differential equations

$$\theta_{i+1} - \theta_i = \theta_i - \theta_{i-1} + 4b, \ i = 1, \cdots, n$$

where $\theta_0 = \inf \mathrm{Supp}(F)$ and $\theta_n = \sup \mathrm{Supp}(F)$. Applying this result, $\bar{\theta}$ is decided by the following system of equations

$$\begin{aligned}
\theta_1 &= \bar{\theta} + 4b \\
\theta_2 &= 2\theta_1 - \bar{\theta} + 4b \\
&\vdots \\
\theta_n &= 2\theta_{n-1} - \theta_{n-2} + 4b, \text{ where } \theta_n = 1 \text{ and } \theta_0 = \bar{\theta}.
\end{aligned}$$

In short,

$$\theta_k = 4b \sum_{i=1}^{k} i + \bar{\theta}, \ k \leq n.$$

For example, if $n = 2$, $\bar{\theta} = 1 - 12b$, $\theta_1 = 1 - 8b$ and $\theta_2 = 1$ constitutes a partition equilibrium of the game starting from T, and the whole revelation process described above[7] is incentive compatible with the sender with $b < \frac{1}{12}$.

Now, suppose that the sender follows the revelation rule specified above, and suppose that time $t < T$ has arrived without receiving m_0. The receiver faces a dynamic choice problem whether to stop and choose now or to continue. If he chooses to stop and take a guess, then he's best guess is $\frac{1}{2}(\frac{t}{\kappa} + 1)$. As a result, he

[5] Crawford and Sobel (1982) says only finitely many messages and actions are played in equilibrium

[6] Since the sender with $\bar{\theta}$ is the lowest type sender among those of who plays a static cheap talk game at T, his most preferred action should be the lowest action, a_1, in equilibrium.

[7] Sender sequentially reveals his type until T, and plays a static cheap talk game at T.

expects

$$g(t)=\int_{t/\kappa}^{1}\left[-\left(\theta-\frac{1}{2}(\frac{t}{\kappa}+1)\right)^2+1\right]\frac{1}{1-t/\kappa}d\theta$$

whose value is $g(t)=1-\frac{(\kappa-t)^2}{12k^2}$. On the other hand, if he receives more messages for Δ period of time, he expects

$$\frac{\kappa-t-\Delta}{\kappa-t}e^{-r\Delta}v\left(t+\Delta\middle|\theta\geq\frac{t+\Delta}{\kappa}\right)+\frac{\Delta}{\kappa-t}\int_{t}^{t+\Delta}e^{-r(\tau-t)}\frac{1}{\Delta}d\tau.$$

where $v(t|\theta\geq\theta')$ denotes the receiver's continuation value at time t when he believes that θ lies on $[\theta',1]$ uniformly. The first term represents the utility the receiver expects when m_1 is not received until $t+\Delta$, and the second term is his expected utility if m_1 is sent after t and before $t+\Delta$ so that he could match his action to the state. Thus, his dynamic programming problem is as follows:

$$v\left(t\middle|\theta\geq\frac{t}{\kappa}\right)=\max\left\{1-\frac{(\kappa-t)^2}{12\kappa^2},\ \frac{\kappa-t-\Delta}{\kappa-t}e^{-r\Delta}v\left(t+\Delta\middle|\theta\geq\frac{t+\Delta}{\kappa}\right)+\frac{1}{r(\kappa-t)}[1-e^{-r\Delta}]\right\}$$

with a terminal condition

$$v(T|\theta\geq\bar{\theta})=v_T,$$

v_T denotes the expected value from cheap talk game at T. As the value function is solely a function of t, we use an artificial value function $V(t)$ to solve the problem, where $V(t)=(\kappa-t)v(t|\kappa\theta\geq t)$. Using this artificial value function, we solve

$$V(t)=\max\left\{(\kappa-t)-\frac{(\kappa-t)^3}{12\kappa^2},\ e^{-r\Delta}V(t+\Delta)+\frac{1}{r}(1-e^{-r\Delta})\right\}$$

The receiver's incentive compatibility is satisfied *if and only if* the second term in max operator is always greater than or equal to the first term. Thus, we have $V(t)=e^{-r\Delta}V(t+\Delta)+\frac{1}{r}(1-e^{-r\Delta})$ in equilibrium. By taking $\Delta\to 0$, we have the following ordinary differential equation[8] with a boundary condition $V(T)=(\kappa-T)v_T$:

$$V'(t)-rV(t)+1=0.$$

The solution to this equation is

$$V(t)=\left[(\kappa-T)v_T-\frac{1}{r}\right]e^{-r(T-t)}+\frac{1}{r}$$

[8] It is because $e^{-r\Delta}\to(1-r\Delta)$ as $\Delta\to 0$.

Thus, we have

$$v(t|\kappa\theta \geq t) = \left[\frac{\kappa(1-\bar{\theta})}{\kappa - t} v_T - \frac{1}{r(\kappa - t)}\right] e^{-r(\kappa\bar{\theta} - t)} + \frac{1}{r(\kappa - t)}.$$

To check the receiver's incentive not to deviate, we need to verify that $v(t|\kappa\theta \geq t)$ indeed is greater than or equal to $g(t)$ for $t \in [0, T]$. Using representation of θ_i in terms of b, v_T is

$$\begin{aligned}
v_T^n &= (1-\bar{\theta})^{-1} \sum_{j=1}^{n} \int_{\theta_{j-1}}^{\theta_j} 1 - (\theta - \frac{\theta_j + \theta_{j-1}}{2})^2 d\theta \\
&= \frac{1}{4b\sum_{j=1}^{n} j} \sum_{j=1}^{n} \int_{\theta_{j-1}}^{\theta_j} 1 - (\theta - \frac{\theta_j + \theta_{j-1}}{2})^2 d\theta \\
&= 1 - \frac{4b^2 \sum_{i=1}^{n} i^3}{3\sum_{i=0}^{n} i} \\
&= 1 - \frac{4b^2 \sum_{i=1}^{n} i}{3} \\
&= 1 - \frac{2b^2 n(n+1)}{3}
\end{aligned}$$

and

$$v^n(t|\kappa\theta \geq t) = \left[\frac{\kappa(1-\bar{\theta})}{\kappa - t}\left[1 - \frac{2b^2 n(n+1)}{3}\right] - \frac{1}{r(\kappa - t)}\right] e^{-r(\kappa\bar{\theta} - t)} + \frac{1}{r(\kappa - t)}.$$

Note: $(\sum_{i=1}^{n} i)^2 = \sum_{i=1}^{n} i^3$.

Lemma 1. $g(t) < \frac{1}{r(\kappa - t)}$, $\forall t < \kappa$.

Proof. Let $t = \alpha\kappa$, $\alpha \in [0, 1)$.

$$\begin{aligned}
\frac{1}{r(\kappa - \alpha\kappa)} - g(\alpha\kappa) &= \frac{1}{r\kappa(1-\alpha)} - 1 + \frac{\kappa^2(1-\alpha)^2}{12\kappa^2} \\
&= \frac{1-b^2}{2b(1-\alpha)} - 1 + \frac{(1-\alpha)^2}{12} \\
&= \frac{1}{12b(1-\alpha)}\left[6 - 6b^2 - 12b(1-\alpha) + b(1-\alpha)^3\right] \\
&\geq \frac{1}{12b(1-\alpha)}\left[6 - 6b^2 - 11b\right] \\
&> 0
\end{aligned}$$

The last inequality holds since $b < \frac{1}{12}$. ■

Thus, if r is small, then g is dominated by v.

3.3. How much to compensate

Remark. If $v^T = g(T)$, then there is $\epsilon > 0$ s.t. $v(T-\epsilon) < g(T-\epsilon)$.

Proof. If $n = 1$, $\bar{\theta} = 1 - 4b$ and $T = \kappa(1-4b)$. Derivative of v at T is

$$\begin{aligned} v'(T) &= \frac{g(T)}{\kappa - T} + rg(T) - \frac{1}{\kappa - T} \\ &= -\frac{b}{3\kappa} + r - \frac{4rb^2}{3} \\ &= r\Big[-\frac{1-b^2}{6} + 1 - \frac{4b^2}{3}\Big] = \frac{r}{6}[5 - 7b^2] \end{aligned}$$

and the derivative of g at T is

$$\frac{\kappa - T}{6\kappa^2} = \frac{2b}{3\kappa} = \frac{r(1-b^2)}{3}.$$

As $b < \frac{1}{12}$, $v'(T) > g'(T)$. ∎

Thus, the receiver's payoff at the end of truthful revelation should be greater than what he expects from a babbling equilibrium. This completes the proof. ∎

3.4. Value of v^n when we change n

$Ev^n(t|\kappa\theta \geq t) = v^n(0|\kappa\theta \geq 0)$, and

$$\begin{aligned} v^n(0|\kappa\theta \geq 0) &= \Big[\frac{\kappa(1-\bar{\theta})}{\kappa}\Big(1 - \frac{4b^2\sum_{i=1}^n i}{3}\Big) - \frac{1}{r\kappa}\Big]e^{-r\kappa\bar{\theta}} + \frac{1}{r\kappa} \\ &= \Big[4b(\sum_{i=1}^n i)\Big(1 - \frac{4b^2\sum_{i=1}^n i}{3}\Big) - \frac{1-b^2}{2b}\Big]e^{-\frac{2b(1-4b\sum_{i=1}^n i)}{1-b^2}} + \frac{1-b^2}{2b} \\ &= \Big[\frac{6bn(n+1) - 4b^3n^2(n+1)^2}{3}\Big]e^{-\frac{2b(1-2bn(n+1))}{1-b^2}} + \frac{1-b^2}{2b}\Big[1 - e^{-\frac{2b(1-2bn(n+1))}{1-b^2}}\Big] \\ &= \Big[\frac{6bn(n+1) - 4b^3n^2(n+1)^2}{3} - \frac{1-b^2}{2b}\Big]e^{-\frac{2b(1-2bn(n+1))}{1-b^2}} + \frac{1-b^2}{2b} \end{aligned}$$

Lemma 2. $v^n(0|\kappa\theta \geq 0)$ increases in n.

Proof.

$$\begin{aligned} 6be^{\frac{2b}{1-b^2}}\frac{dv^n}{dn} =& \Big[12b^2(n+1) + 12b^2n - 16b^4n(n+1)^2 - 16b^4n^2(n+1)\Big]e^{\frac{4b^2n(n+1)}{1-b^2}} \\ &+ \Big[\frac{4b^2(n+1) + 4b^2n}{1-b^2}\Big]\Big[12b^2n(n+1) - 8b^4n^2(n+1)^2 - 3 + 3b^2\Big]e^{\frac{4b^2n(n+1)}{1-b^2}} \end{aligned}$$

Here,

$$sgn\left(\frac{dv^n}{dn}\right) = sgnf,\ f = 12b^2(2n+1) - 16b^4(2n^3+3n^2+n) + \frac{4b^2(2n+1)}{1-b^2}\Big[12b^2n(n+1) - 8b^4n^2(n+1)^2 - 3 + 3b^2$$

$$\begin{aligned} f =& 4b^2(2n+1)\Big[-4b^2(n+1) + \frac{12b^2n(n+1) - 8b^4n^2(n+1)^2}{1-b^2}\Big] \\ =& \frac{16b^4(n+1)(2n+1)}{1-b^2}\Big[-1 + b^2 + 3n - 2b^2n^2(n+1)\Big] \\ \geq& \frac{16b^4(n+1)(2n+1)}{1-b^2}\Big[-1 + b^2 + 3n - \frac{n}{12}\Big]. \end{aligned}$$

The last inequality is because $\bar{\theta} = 1 - 2bn(n+1) \geq 0$ and $b \leq \frac{1}{12}$ imply $b^2n(n+1) \leq \frac{1}{24}$. Since $-1 + b^2 + 3n - \frac{n}{12} > 0$, we conclude that it indeed increases in n. ■

3.5. What is Max n possible for each b?

Let $n(b)$ be the maximum n possible with the revelation rule. It is decided by

$$2bn(n+1) \leq 1.$$

It should be $1 \leq n \leq \frac{-1+\sqrt{1+\frac{2}{b}}}{2}$. Thus,

$$n(b) = \llcorner \frac{-1+\sqrt{1+\frac{2}{b}}}{2} \lrcorner,$$

where $\llcorner x \lrcorner$ is the largest integer smaller than or equal to x. Recall that in CS, $N(b) = \ulcorner \frac{-1+\sqrt{1+\frac{2}{b}}}{2} \urcorner$. Thus, it is either $n(b) = N(b)$ or $n(b) = N(b) - 1$. e.g., $b = \frac{1}{12}$, $n(b) = N(b) = 2$.

3.6. Welfare comparison

Note that if $n(b) = N(b)$, then there's no delay in decision, and the resulting utilities are the same(same partition induced).

(i) Receiver's ex-ante utility from dynamic revelation (num of partition n)

$$\left[\frac{6bn(n+1) - 4b^3n^2(n+1)^2}{3} - \frac{1-b^2}{2b}\right]e^{-\frac{2b(1-2bn(n+1))}{1-b^2}} + \frac{1-b^2}{2b}$$

(ii) Receiver's ex-ante utility from CS's most informative equilibrium (num of partition $n+1$)

$$1 - \frac{1}{12(n+1)^2} - \frac{b^2((n+1)^2-1)}{3}$$

Sender's utility:

$$
\begin{aligned}
v_T^S &= \frac{1}{1-\bar{\theta}} \sum_{i=1}^{n} \int_{\theta_{i-1}}^{\theta_i} 1 - (\theta + b - \frac{\theta_i + \theta_{i-1}}{2})^2 d\theta \\
&= \frac{1}{1-\bar{\theta}} \sum_{i=1}^{n} \Big[4bi - \frac{b^3(2i+1)^3}{3} - \frac{b^3(2i-1)^3}{3} \Big] \\
&= 1 - \frac{2b^2 n(n+1)}{3} - b^2 \\
&= v_T - b^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v^S(t=0) &= \int_0^{\bar{\theta}} (1-b^2) e^{-r\kappa\theta} d\theta + (1-\bar{\theta}) v_T^S e^{-r\kappa\bar{\theta}} \\
&= \Big[-(1-b^2)\frac{1-b^2}{2b} + (1-\bar{\theta})(v_T - b^2) \Big] e^{-r\kappa\bar{\theta}} + (1-b^2)\frac{1-b^2}{2b} \\
&= v(0) - b^2 \Big[\frac{1-b^2}{2b} + (1-\bar{\theta}) e^{-r\kappa\bar{\theta}} - \frac{1-b^2}{2b} e^{-r\kappa\bar{\theta}} \Big]
\end{aligned}
$$

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